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Commutants and bicommutants of operators of class C_0

BÉLA SZ.-NAGY and CIPRIAN FOIAŞ

Dedicated to P. R. Halmos on his 60th birthday

Introduction

By operator we mean a linear and bounded one. For any operator T on a Hilbert space $\mathfrak{H}$ we consider the following weakly (or equivalently, strongly) closed subalgebras of $\mathscr{B}(\mathfrak{H})$:

$\mathscr{A}_T$: the subalgebra generated by I and T;

$\{T\}'$: the commutant of T;

$\{T\}''$: the bicommutant of T;

$\mathscr{L}_T$: the subalgebra consisting of those $X\in\{T\}'$ for which $\operatorname{Lat} X\supset\operatorname{Lat} T$ (i.e. X leaves invariant every subspace of $\mathfrak{H}$ invariant for T).

If T is a completely non-unitary contraction on $\mathfrak{H}$ we also define:

$\mathscr{N}_T$: the set of operators on $\mathfrak{H}$ which admit a representation $X=v(T)^{-1}u(T)$ with functions $u, v\in H^\infty$ such that $v(T)$ is a quasi-affinity (i.e. an operator with zero kernel and dense range).

From this definition it readily follows:

$$\mathscr{N}_T\subset\{T\}'', \quad cf.\ [H], \text{ Chapter IV.} \tag{0}$$

We shall consider operators T of class C_0, i.e. completely non-unitary contractions such that $w(T)=0$ for some inner function w; among these functions w there is a minimal one, denoted by m_T. For $T\in C_0$ and $v\in H^\infty$ the operator $v(T)$ is a quasi-affinity if and only if $v\wedge m_T=1$ (i.e. if v and m_T have no non-constant inner divisor); *cf.* [H], Proposition III. 4.7.

For $T\in C_0$ we have equality in (0), i.e.

$$\mathscr{N}_T=\{T\}'' \quad \text{for} \quad T\in C_0. \tag{1}$$

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This was proved in [2] if the underlying space $\mathfrak{H}$ is separable, by using the "Jordan model" of operators of class C_0. A subsequent extension of the Jordan model to the non-separable case, given in [3], yields, by the same proof, the validity of (1) for non-separable $\mathfrak{H}$ also.

In Sections 1 and 2 of the present paper we shall prove the inclusions

(2) $$\mathcal{N}_T \subset \mathcal{A}_T \quad \text{for} \quad T \in C_0,$$

(3) $$\mathcal{L}_T \subset \mathcal{N}_T \quad \text{for} \quad T \in C_0.$$

As a consequence of (1), (2), (3), and of the trivial inclusion $\mathcal{A}_T \subset \mathcal{L}_T$ we deduce

$$\{T\}'' = \mathcal{N}_T \subset \mathcal{A}_T \subset \mathcal{L}_T \subset \mathcal{N}_T = \{T\}'' \quad \text{for} \quad T \in C_0.$$

So we establish the following:

Theorem. *For any operator T of class C_0 we have*

$$\mathcal{A}_T = \mathcal{L}_T = \{T\}'' = \mathcal{N}_T.$$

For operators T of class C_0 with *finite* defect indices (classes $C_0(N)$; $N = 1, 2, \ldots$) these results were proved in the recent paper [4] by Wu (Theorems 3.2 and 3.3). It was this paper that suggested the present investigation. The proofs we are going to give for the general case employ quite different arguments as those in [4].

1. Proof of $\mathcal{N}_T \subset \mathcal{A}_T$

Let $T \in C_0$ on $\mathfrak{H}$. Suppose there is an $X \in \mathcal{N}_T$ which is not contained in $\mathcal{A}_T$. This means that there exist $h_1, \ldots, h_r \in \mathfrak{H}$ and $\varepsilon > 0$ such that

(1.1) $$\sum_{j=1}^{r} \|Xh_j - p(T)h_j\|^2 \geqq \varepsilon^2 \quad \text{for all polynomials } p.$$

Setting

$$\mathbf{H} = \bigoplus_1^r \mathfrak{H}, \quad \mathbf{T} = \bigoplus_1^r T, \quad \mathbf{X} = \bigoplus_1^r X, \quad \mathbf{h} = \bigoplus_1^r h_j,$$

(1.1) can also be expressed as

(1.2) $$\|\mathbf{X}\mathbf{h} - p(\mathbf{T})\mathbf{h}\| \geqq \varepsilon \quad \text{for all polynomials } p.$$

As $X \in \mathcal{N}_T$ there exist $u, v \in H^\infty$ such that $v \wedge m_T = 1$, $v(T)X = u(T)$, and hence,

(1.3) $$v(\mathbf{T})\mathbf{X} = u(\mathbf{T}).$$

Denote by $\mathbf{H}_\mathbf{h}$ the cyclic subspace for $\mathbf{T}$ generated by $\mathbf{h}$ and define

(1.4) $$\mathbf{K} = \{\mathbf{k} \in \mathbf{H} : v(\mathbf{T})\mathbf{k} \in \mathbf{H}_\mathbf{h}\}.$$

Clearly, $\mathbf{K}$ is invariant for $\mathbf{T}$ and $\mathbf{T}_0=\mathbf{T}|\mathbf{K}$ is of class C_0. Its minimal function is a divisor of $m_{\mathbf{T}}$ $(=m_T)$ so we also have $v\wedge m_{\mathbf{T}_0}=1$. Thus, $v(\mathbf{T}_0)$ is a quasi-affinity on $\mathbf{K}$ and so it has dense range in $\mathbf{K}$. As by definition (1.4)

$$v(\mathbf{T}_0)\mathbf{K}=v(\mathbf{T})\mathbf{K}\subset\mathbf{H}_{\mathbf{h}}$$

we infer that

$$\mathbf{K}\subset\mathbf{H}_{\mathbf{h}}. \tag{1.5}$$

Now, by (1.3) we have $v(\mathbf{T})\mathbf{X}\mathbf{h}=u(\mathbf{T})\mathbf{h}\in\mathbf{H}_{\mathbf{h}}$, and therefore $\mathbf{X}\mathbf{h}\in\mathbf{K}$; thus, by (1.5),

$$\mathbf{X}\mathbf{h}\in\mathbf{H}_{\mathbf{h}}.$$

This implies that there is a polynomial p such that

$$\|\mathbf{X}\mathbf{h}-p(\mathbf{T})\mathbf{h}\|<\varepsilon.$$

This contradicts (1.2), and hence achieves the proof.

2. Proof of $\mathscr{L}_T\subset\mathscr{N}_T$

Let $T\in C_0$ on $\mathfrak{H}$. By [2], Proposition 2, we have

$$T\succ S(m)\oplus G,$$

where $m=m_T$ and G is the restriction of T to some invariant subspace $\mathfrak{G}$, i.e. there exists a quasi-affinity

$$A:\mathfrak{H}(m)\oplus\mathfrak{G}\to\mathfrak{H}$$

such that

$$TA=A\big(S(m)\oplus G\big). \tag{2.1}$$

Here, as usual, $S(m)$ denotes the compression of the canonical shift on H^2 to the subspace $\mathfrak{H}(m)=H^2\ominus mH^2$.

Consider the cyclic vector e for $S(m)$, given by $e=1-\overline{m(0)}m$, and an arbitrarily chosen vector $g\in\mathfrak{G}$, and set

$$h_t=A\big((1-t)e\oplus tg\big), \tag{2.2}$$

t being a numerical parameter to be fixed later. Further, set

$$\mathfrak{H}_t=\bigvee_{n\geqq0}T^nh_t,\quad T_t=T|\mathfrak{H}_t,\quad\text{and}\quad m_t=m_{T_t}.$$

From (2.1) and (2.2) we deduce

$$w(T)h_0=A\big(w(S(m))e\oplus0\big)\quad\text{for all}\quad w\in H^\infty;$$

hence T_0 has the same minimal function as $S(m)$, i.e. $m_0=m=m_T$.

While it may happen that m_1 is a proper inner divisor of m_T, it follows from a lemma due to M. SHERMAN that the values t for which m_t is a proper divisor of m_T are exceptional, that is, countable many; *cf.* [1]. Let τ be a non-exceptional value of t, different from 0 and 1; thus $m_\tau = m_T$, $0 \neq \tau \neq 1$.

Let $X \in \mathscr{L}_T$. Then $X\mathfrak{H}_t \subset \mathfrak{H}_t$ and $X|\mathfrak{H}_t \in \{T_t\}'$, for all t. Since T_t is a C_0 class operator with cyclic vector h_t, every operator in its commutant is a function of T_t of the "Nevanlinna class" $\mathscr{N}_{T_t}$ (*cf.* [H], Chapter IV, and [1], Théorème 2). Thus there exist functions $u_t, v_t \in H^\infty$ such that

$$v_t \wedge m_t = 1 \quad \text{and} \quad v_t(T)Xh_t = u_t(T)h_t; \tag{2.3}$$

in particular,

$$v_0 \wedge m_T = 1, \quad v_\tau \wedge m_T = 1. \tag{2.4}$$

Set

$$X' = v_0(T)X - u_0(T); \tag{2.5}$$

X' also belongs to $\mathscr{L}_T$ and by (2.3) we have

$$X'h_0 = 0 \quad \text{and} \quad v_t(T)X'h_t = u_t'(T)h_t \quad \text{for} \quad u_t' = v_0 u_t - u_0 v_t. \tag{2.6}$$

Hence, $X'h_t = X'((1-t)h_0 + th_1) = tX'h_1$ and

$$v_t(T)v_1(T)X'h_t = \begin{cases} v_t(T)\cdot tu_1'(T)h_1 = t(v_t u_1')(T)h_1 \\ v_1(T)v_t(T)X'h_t = v_1(T)u_t'(T)h_t = (v_1u_t')(T)((1-t)h_0 + th_1) \end{cases}$$

so we have

$$(1-t)(v_1u_t')(T)h_0 = t(v_tu_1' - v_1u_t')(T)h_1.$$

By (2.1) and (2.2), and since A is injective, this implies

$$(1-t)\cdot(v_1u_t')(S(m))e \oplus 0 = 0 \oplus t\cdot(v_tu_1' - v_1u_t')(G)g;$$

so we have for any $t \neq 0$, 1, and in particular for $t = \tau$:

$$(v_1u_\tau')(S(m))e = 0, \quad (v_\tau u_1' - v_1u_\tau')(G)g = 0. \tag{2.7}$$

The first equation (2.7) implies $v_1u_\tau' \in mH^\infty$. Since $m_G | m$ we infer $(v_1u_\tau')(G) = 0$. Comparing this with the second equation (2.7) we deduce

$$v_\tau(G)u_1'(G)g = (v_\tau u_1')(G)g = 0.$$

On account of (2.4), $v_\tau(T)$ is a quasi-affinity so its restriction $v_\tau(G)$ is injective; thus $u_1'(G)g = 0$. Hence,

$$u_1'(T)h_1 = u_1'(T)A(0 \oplus g) = Au_1'(S(m) \oplus G)(0 \oplus g) = A(0 \oplus u_1(G)g) = 0,$$

and therefore, by (2.6), $v_1(T)X'h_1 = 0$. Now, the subspace $\mathfrak{H}_1$ being invariant for T is also invariant for X'; thus $X'h_1 \in \mathfrak{H}_1$. But $v_1 \wedge m_1 = 1$ by (2.3), and thus $v_1(T_1) = v_1(T)|\mathfrak{H}_1$ is a quasi-affinity on $\mathfrak{H}_1$, and in particular injective, so we conclude $X'h_1 = 0$.

Combining this result with the equation $X'h_0=0$, see (2.6), and recalling that by its definition (2.5) the operator X' is independent of the choice of g in $\mathfrak{G}$ we readily conclude that $X'A=0$, $X'=0$, and therefore

$$v_0(T)X-u_0(T)=0 \quad (v_0 \wedge m_T=1),$$

that is, $X \in \mathcal{N}_T$.

This concludes the proof.

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B. SZ.-NAGY
BOLYAI INSTITUTE
ARADI VÉRTANÚK TERE 1
6720 SZEGED, HUNGARY

C. FOIAŞ
FACULTY OF MATH., UNIV. BUCHAREST
STR. ACADEMIEI 14
BUCHAREST, ROMANIA